CONTINUOUS REDUCIBILITY AND DIMENSION OF METRIC SPACES

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Abstract. If (X, d) is a Polish metric space of dimension 0, then by Wadge's lemma, no more than two Borel subsets of X can be incomparable with respect to continuous reducibility. In contrast, our main result shows that for any metric space (X, d) of positive dimension, there are uncountably many Borel subsets of (X, d) that are pairwise incomparable with respect to continuous reducibility.

The reducibility that is given by the collection of continuous functions on a topological space (X, τ) is called the *Wadge quasi-order* for (X, τ). We further show that this quasi-order, restricted to the Borel subsets of a Polish space (X, τ), is a *well-quasiorder (wqo)* if and only if (X, τ) has dimension 0, as an application of the main result.

Moreover, we give further examples of applications of the technique, which is based on a construction of graph colorings.

Contents

1. Introduction

The *Wadge quasi-order* on the subsets of the Baire space is an important notion that is used to fit definable sets into a hierarchy of complexity. Its importance comes from the fact that it defines the finest known hierarchy on various classes of definable subsets of the Baire space. For instance, it refines the difference hierarchy, the Borel hierarchy and the projective hierarchy.

The structure of the Wadge quasi-order on the Baire space and its closed subsets has therefore been an object of intensive research (see e.g. [Wad12, Lou83, And06]). Moreover, many structural results could even be extended to many other classes of functions (see e.g. [AM03, MR09, MR10a, MR10b, MR14, MRS14]).

In this paper, we study the Wadge quasi-order on the class of Borel subsets of arbitrary metric spaces. By a Borel subset of a topological space, we mean an element of the least σ-algebra that contains the open sets. We define the Wadge quasi-order on arbitrary topological spaces as follows.

Definition 1.1. Suppose that (X, τ) is a topological space and A, B are subsets of X.

(a) A is *reducible* to B ($A \leq_{(X,\tau)} B$) if $A = f^{-1}[B]$ for some continuous map $f : X \to X$.
(b) A, B are *equivalent* ($A \sim_{(X,\tau)} B$) if $A \leq_{(X,\tau)} B$ and $B \leq_{(X,\tau)} A$.
(c) A, B are *comparable* if $A \leq_{(X,\tau)} B$ or $B \leq_{(X,\tau)} A$, and otherwise *incomparable*.

The following principle states a further important property of these quasi-orders.

Definition 1.2. Assuming that (X, τ) is a topological space, the *semi-linear ordering principle* $\mathsf{SLO}_{(X,\tau)}$ states that for all Borel subsets A, B of X, A is reducible to B, or B is reducible to A.

This paper supersedes an earlier preprint, where the main result of this paper was proved for Polish spaces, by an unrelated proof.

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It is easy to see that $\mathsf{SLO}_{(X,\tau)}$ implies that no more than two Borel subsets of X can be incomparable, Moreover, Woodin observed that $\mathsf{SLO}_{\mathbb{R}}$ fails (see [Woo10, Remark 9.26], [And07, Example 3]) and thus it is natural to ask the following question.

Question 1.3. *Under which conditions on a Polish space (X,τ) does $\mathsf{SLO}_{(X,\tau)}$ fail?*

The principle SLO holds for any Polish space that is homeomorphic to a closed subset of the Baire space, since the proof of Wadge's lemma (see e.g. [And07, Section 2.3]) works for these spaces. Moreover, these Polish space are known to be exactly the ones with dimension 0, which is defined as follows.

Definition 1.4. A topological space (X,τ) has *dimension* 0 if for every x in X and every open set U subset of X containing x, there is a subset of U containing x that is both open and closed. Moreover, the space has *positive dimension* if it does not have dimension 0.

This is usually denoted by having *small inductive dimension* 0, but note that for separable metric spaces, it is equivalent to the condition that other standard notions of dimension, such as the large inductive dimension 0 or the Lebesgue covering dimension, are equal to 0 by [Eng89, Theorem 7.3.2]. However, not every totally disconnected Polish space has dimension 0, since for instance the complete Erdös space [DvM09] has the former property, but not the latter.

The following is the main result of this paper, which is proved in Theorem 2.15 below, answering Question 1.3.

Theorem 1.5. *For any metric space (X,d) of positive dimension, there are uncountably many Borel subsets of (X,d) that are pairwise incomparable with respect to continuous reducibility.*

As an application of the main result, we will characterize the Wadge order on Polish spaces by the following notion, which is important in the theory of quasi-orders (see e.g. [CP14]).

Definition 1.6. A *well-quasiorder (wqo)* is a quasi-order with the property that there is no infinite strictly decreasing sequence and no infinite set of pairwise incomparable elements.

The next characterization is proved in Theorem 2.16 below.

Theorem 1.7. *Suppose that (X,d) is a Polish metric space. Then the following conditions are equivalent.*

(a) X has dimension 0.
(b) $\mathsf{SLO}_{(X,\tau)}$ holds.
(c) The Wadge order on the Borel subsets of X is a well-quasiorder.
(d) There are at most two pairwise incomparable Borel subsets of X.
(e) There are at most countably many pairwise incomparable Borel subsets of X.

It is worthwhile to mention that Pequignot defines an alternative quasi-order on the subsets of an arbitrary Polish space [Peq15]. Moreover, his notion is more natural in the sense that it always satisfes the SLO principle.

We will further prove the following variant of the main result in Theorem 4.1 below.

Theorem 1.8. *Suppose that (X,d) is a locally compact metric space of positive dimension. Then there is a (definable) injective map that takes sets of reals to subsets of X in such a way that these subsets are pairwise incomparable with respect to continuous reducibility.*

This paper has the following structure. We will prove Theorem 1.5 and Theorem 1.7 in Section 2, but postpone the proof of some auxiliary results to Section 3. Moreover, Section 4 contains the proof of Theorem 1.8, and some further remarks on the proofs can be found in Section 5.

2. Incomparable Borel sets

In this section, we will prove Theorem 1.5, except for some technical steps that are postponed to the next sections.

We assume that (X,d) is a metric space of positive dimension. By the definition, there is some $x^* \in X$ with no neighborhood base at x^* that consists of sets that are both open and closed. We will fix such an element x^* for the remainder of this section.

For any $x \in X$, we will denote the open ball with radius r around x by

$$B_r(x) = \{y \in X \mid d(x,y) < r\}.$$

By the choice of x^*, there is some $r^* > 0$ such that no closed-open neighborhood of x^* is completely contained in $B_r(x^*)$. We will fix r^* for the remainder of this section. Moreover, let $X^* = B_r(x^*)$ and fix a strictly increasing sequence $\mathbf{r} = \langle r_n \mid n \geq 0\rangle$ with $r_0 = 0$ and $\sup_{n\geq 0} r_n = r^*$.

Definition 2.1. If $A \subseteq \mathbb{R}_{\geq 0}$, let

$$C_A = \{x \in X^* \mid d(x,x^*) \in A\}$$

if $\inf(A) > 0$ and

$$C_A = \{x \in X^* \mid d(x,x^*) \in A\} \cup \{x^*\}$$

otherwise.

We will always use the notation $\mathbf{m} = \langle m_i \mid i \geq 0\rangle$ and $\mathbf{n} = \langle n_i \mid i \geq 0\rangle$ to denote strictly increasing sequences of natural numbers beginning with 0. Moreover, we will frequently use the following notation.

Definition 2.2. Suppose that $\mathbf{n}$ is as above. Letting $[n_i, n_{i+1})$ denote the interval in $\mathbb{N}$, we define the following sets of natural numbers.

(a) $\mathrm{Even}_{\mathbf{n}} = \bigcup_{i\in\mathbb{N} \text{ is even}} [n_i, n_{i+1})$,
(b) $\mathrm{Odd}_{\mathbf{n}} = \bigcup_{i\in\mathbb{N} \text{ is odd}} [n_i, n_{i+1})$.

In the next definition, we define sets $D_{\mathbf{n}}$ for sequences $\mathbf{n}$ as above. These sets will later be shown to be incomparable under appropriate assumptions.

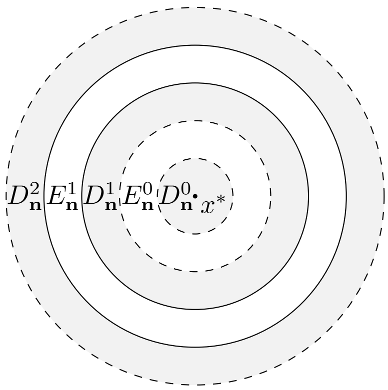

FIGURE 1. A diagram of $D_{\mathbf{n}}$ (shaded), assuming that $n_0 = 0$, $n_1 = 1$ and $n_2 = 2$. The solid lines denote elements of $D_{\mathbf{n}}$ and the dotted lines denote elements of $E_{\mathbf{n}}$.

Definition 2.3. Suppose that $\mathbf{n}$ is as above.

(a) (i) Let s denote the string of symbols $2j, 2j+1$.

$$D^j_{\mathbf{n}} = \begin{cases} C_{[s)} & \text{if } j \in \mathrm{Even}_{\mathbf{n}}, \\ C_{(s]} & \text{if } j \in \mathrm{Odd}_{\mathbf{n}}, \end{cases}$$

(ii) $D_{\mathbf{n}} = \bigcup_{j\geq 0} D^j_{\mathbf{n}}$,

(b) (i) Let s denote the string of symbols $2j+1, 2j+2$ and let t denote the ordered pair $(j, j+1)$.

$$E^j_{\mathbf{n}} = \begin{cases} C_{[s)} & \text{if } t \in \mathrm{Even}_{\mathbf{n}} \times \mathrm{Even}_{\mathbf{n}}, \\ C_{[s]} & \text{if } t \in \mathrm{Even}_{\mathbf{n}} \times \mathrm{Odd}_{\mathbf{n}}, \\ C_{(s)} & \text{if } t \in \mathrm{Odd}_{\mathbf{n}} \times \mathrm{Even}_{\mathbf{n}}, \\ C_{(s]} & \text{if } t \in \mathrm{Odd}_{\mathbf{n}} \times \mathrm{Odd}_{\mathbf{n}}, \end{cases}$$

(ii) $E_{\mathbf{n}} = \bigcup_{j\geq 0} E^j_{\mathbf{n}}$.

The sets $E_{\mathbf{n}}^j$ are chosen to partition the complement of $D_{\mathbf{n}}$ in X^*, and thus the sets $D_{\mathbf{n}}$, $E_{\mathbf{n}}$ partition X^*, as illustrated in Figure 1.

The idea for the proof of Theorem 1.5 in the remainder of this section is as follows. We will associate certain graphs to the sets $D_{\mathbf{n}}$ for various sequences $\mathbf{n}$ as above. Then, we will show that the existence of a continuous function $F\colon X \to X$ with $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$ implies the existence of certain maps associated to these graphs, and moreover, that such maps cannot exist if $\mathbf{m}$, $\mathbf{n}$ are sufficiently different.

The combinatorics in the following proofs reflect the fact that it does not follow from the existence of such a reduction F that $\mathbf{m}$ and $\mathbf{n}$ are equal, as can be seen from the examples in Figure 2.

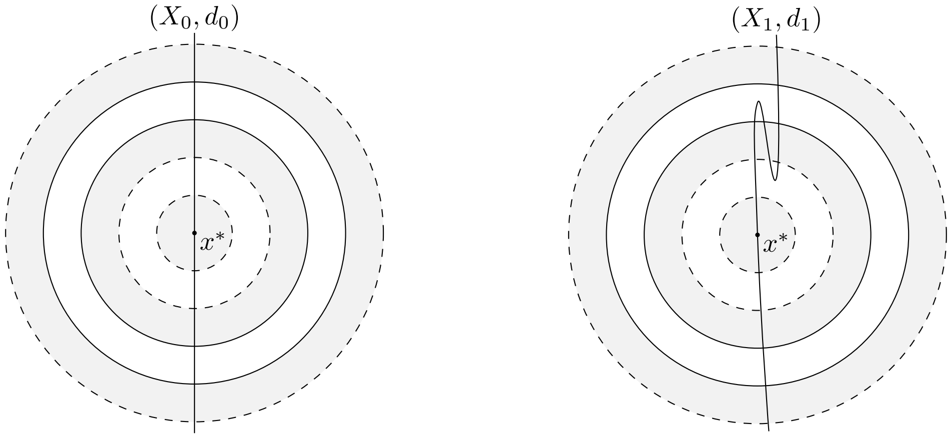

FIGURE 2. Diagrams of $D_{\mathbf{n}}$ (shaded) for two one-dimensional subspaces of $\mathbb{R}^2$.

In (undirected) graphs, we will identify the edges, i.e. 2-element sets of vertices, with ordered pairs.

Definition 2.4. A *colored graph* consists of a graph G with vertex set V and edge set E that satisfy the following conditions, together with a coloring c_G in the colors 0 and 1 that is defined both on the vertices and the edges.

(a) V is a (finite or infinite) interval in $\mathbb{Z}$ of size at least 2.
(b) E consists of the pairs of successive vertices in V.

We will write $c_G(i, i+1)$ for the color of the edge $(i, i+1)$ and thereby omit the additional brackets.

Definition 2.5. Suppose that G and H are colored graphs.

(a) A *reduction f from G to H* is a function defined on both the vertices and the edges of G that satisfies the following conditions. The vertices of G are mapped to vertices of H, the edges of G are mapped to edges of H and the map preserves colors in both cases. Moreover, if $(i, i+1)$ is an edge in G, then $f(i)$ and $f(i+1)$ are end points of the edge $f(i, i+1)$ in H. We will write $\mathrm{ran}(f)$ for the set of vertices that are in the range of f.
(b) An *unfolding of G, H* is a pair $\xi = (f, g)$ such that for some finite colored graph I, f is a reduction from I to G and g is a reduction from I to H. We will write $\mathrm{dom}(\xi)$ for the vertex set of I and let $c_\xi = c_I$.

It follows immediately from the definition of reductions that the image of a reduction f is a finite interval in $\mathbb{Z}$, and for all i with $i, i+1 \in \mathrm{dom}(f)$ and $f(i) \neq f(i+1)$, $f(i, i+1) = (f(i), f(i+1))$.

The followings colored graphs carry information about the sets $D_{\mathbf{n}}$ and $E_{\mathbf{n}}$ defined above. Moreover, the maps between graphs defined below carry information about possible continuous reductions from $D_{\mathbf{m}}$ to $D_{\mathbf{n}}$ for sequences $\mathbf{m}$, $\mathbf{n}$ as above.

Definition 2.6. Suppose that $\mathbf{n}$ is as above. Let $G_{\mathbf{n}}$ denote the colored graph with vertex set $\mathbb{N}$ and the coloring $c_{\mathbf{n}}$ defined on the vertices by

$$c_{\mathbf{n}}(2j) = \begin{cases} 1 & \text{if } j \in \mathrm{Even}_{\mathbf{n}} \\ 0 & \text{if } j \in \mathrm{Odd}_{\mathbf{n}} \end{cases}$$

$$c_{\mathbf{n}}(2j+1) = \begin{cases} 0 & \text{if } j \in \mathrm{Even}_{\mathbf{n}} \\ 1 & \text{if } j \in \mathrm{Odd}_{\mathbf{n}} \end{cases}$$

for $j \geq 0$ and on the edges by

$$c_{\mathbf{n}}(j, j+1) = \begin{cases} 1 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

for $j \geq 0$.

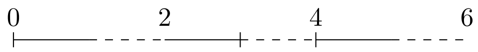

FIGURE 3. A diagram of the graph $G_{\mathbf{n}}$ corresponding to Figures 1 and 2. The solid lines denote edges with color 1 and the dotted lines denote edges with color 0. Moreover, the marked vertices have color 1 and the other ones have color 0.

Definition 2.7. Suppose that $\mathbf{m}$, $\mathbf{n}$ are as above and $\xi = (f, g)$ is an unfolding of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$, where f is a reduction from I to $G_{\mathbf{m}}$ and g is a reduction from I to $G_{\mathbf{n}}$. We define the relation $\sim_{\xi}^{\mathbf{m},\mathbf{n}}$ on pairs of vertices and pairs of edges as follows, and abbreviate it by $\sim_{\xi}$.

(a) If k is a vertex in $G_{\mathbf{m}}$ and l is a vertex in $G_{\mathbf{n}}$, let $k \sim_{\xi} l$ if $f(j) = k$ and $g(j) = l$ for some vertex j in I.
(b) If v is an edge in $G_{\mathbf{m}}$ and w is an edge in $G_{\mathbf{n}}$, let $v \sim_{\xi} w$ if $f(u) = v$ and $g(u) = w$ for some edge u in I.

Definition 2.8. Suppose that $\mathbf{m}$, $\mathbf{n}$ are as above and $F \colon X \to X$ is a continuous map with $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$. Moreover, suppose that $x \in X^*$ and $\xi = (f, g)$ is an unfolding of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$. We say that x *and* ξ *are compatible with respect to* F if one of the following conditions holds.

(a) $(x, F(x)) \in D_{\mathbf{m}}^i \times D_{\mathbf{n}}^j$ for some pair (i, j) with
$$(2i, 2i+1) \sim_{\xi} (2j, 2j+1).$$
(b) $(x, F(x)) \in E_{\mathbf{m}}^i \times E_{\mathbf{n}}^j$ for some pair (i, j) with
$$(2i+1, 2i+2) \sim_{\xi} (2j+1, 2j+2).$$

Let $X_{\xi,F}^{\mathbf{m},\mathbf{n}}$ denote the set of $x \in X^*$ such that x and ξ are compatible with respect to F. We will also call $X_{\xi,F}^{\mathbf{m},\mathbf{n}}$ the *compatibility range* of $\mathbf{m}$, $\mathbf{n}$, ξ and F.

Definition 2.9. Suppose that $\mathbf{m}$, $\mathbf{n}$ and F are as above and $i \geq 0$.

(a) Let $\mathbb{U}_i^{\mathbf{m},\mathbf{n}}$ denote the set of unfoldings ξ with $0 \sim_{\xi}^{\mathbf{m},\mathbf{n}} i$.
(b) Let $X_{i,F}^{\mathbf{m},\mathbf{n}} = \bigcup_{\xi \in \mathbb{U}_i^{\mathbf{m},\mathbf{n}}} X_{\xi,F}^{\mathbf{m},\mathbf{n}}$.

The key to the proof of the main theorem is given by the next two lemmas, whose proofs we postpone to Section 3.1 due to their technical nature.

Lemma 2.10. *For all* $\mathbf{m}$*,* $\mathbf{n}$*,* F *as above and* $i \geq 0$*,* $X_{i,F}^{\mathbf{m},\mathbf{n}}$ *is an open subset of* X^**.*

Lemma 2.11. *For all* $\mathbf{m}$*,* $\mathbf{n}$*,* F *as above and* $i \geq 0$*,* $X_{i,F}^{\mathbf{m},\mathbf{n}}$ *is a relatively closed subset of* X^**.*

Note that it follows from the first lemma that $X_{i,F}^{\mathbf{m},\mathbf{n}}$ is an open subset of the full space X, but in the second lemma, we do not require that $X_{i,F}^{\mathbf{m},\mathbf{n}}$ is a closed subset of X.

Given these lemmas, the remaining step is to show, for the equivalence relation E_{tail} given in the next definition, that whenever $\mathbf{m}$, $\mathbf{n}$ are sequences as above and there is a continuous reduction from $D_{\mathbf{m}}$ to $D_{\mathbf{n}}$, then $\mathbf{m}$, $\mathbf{n}$ are E_{tail}-equivalent.

Definition 2.12. Suppose that $\mathbf{m}$, $\mathbf{n}$ are as above.

(a) We define an equivalence relation E_{tail} on the set of $\mathbf{n}$ as above as follows. The sequences $\mathbf{m}$, $\mathbf{n}$ are E_{tail}-*equivalent* if there is some $i_0 \geq 0$ and some $j \in \mathbb{Z}$ such that $m_i = n_{i+j}$ for all $i \geq i_0$.

(b) Let $\Delta\mathbf{m}$ denote the sequence with values $(\Delta\mathbf{m})_i = m_{i+1} - m_i$ for all $i \geq 0$.

Lemma 2.13. *Suppose that* $\mathbf{m}$*,* $\mathbf{n}$ *are as above such that also* $\Delta\mathbf{m}$*,* $\Delta\mathbf{n}$ *are strictly increasing. Moreover, suppose that*

$$\bigcup_{(f,g)\in\mathbb{U}_l^{\mathbf{m},\mathbf{n}}} \operatorname{ran}(f) = \mathbb{N}$$

for some $l \geq 0$*. Then* $\Delta\mathbf{m}$*,* $\Delta\mathbf{n}$ *are* E_{tail}*-equivalent.*

We postpone the proof of the previous lemma to Section 3.2 due to its technical nature.

Lemma 2.14. *Suppose that* $\mathbf{m}$*,* $\mathbf{n}$ *are as in the previous lemma. If* $D_{\mathbf{m}} \leq_{(X,d)} D_{\mathbf{n}}$*, then* $\Delta\mathbf{m}$*,* $\Delta\mathbf{n}$ *are* E_{tail}*-equivalent.*

Proof. Recall that $x^* \in X$ and $r^* > 0$ were chosen in the beginning of this section such that the open ball $B_{r^*}(x^*)$ contains no neighborhood of x^* that is both open and closed in X. Moreover, $\mathbf{r} = \langle r_n \mid n \in \omega \rangle$ is strictly increasing with $r_0 = 0$, $\sup_n r_n = r^*$ and $X^* = B_{r^*}(x^*)$.

Since $D_{\mathbf{m}} \leq_{(X,d)} D_{\mathbf{n}}$ by the assumption, there is a continuous map $F\colon X \to X$ with $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$. Moreover, it follows from the definition of $D_{\mathbf{m}}$ that $x^* \in D_{\mathbf{m}}$ and therefore $F(x^*) \in D_{\mathbf{n}} \subseteq X^*$. Since the sets $D^j_{\mathbf{n}}$ are pairwise disjoint for different $j \geq 0$, there is a unique $l \geq 0$ with $F(x^*) \in D^l_{\mathbf{n}}$. We can assume that $l \in \mathrm{Even}_{\mathbf{n}}$, since the case that $l \in \mathrm{Odd}_{\mathbf{n}}$ is symmetric.

Claim. $x^* \in X^{\mathbf{m},\mathbf{n}}_{2l,F}$.

Proof. By the definition of $X^{\mathbf{m},\mathbf{n}}_{2l,F}$ (see Definition 2.9), it is sufficient to show that there is some unfolding ξ of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ with $0 \sim^{\mathbf{m},\mathbf{n}}_{\xi,F} 2l$ such that x^* and ξ are compatible with respect to F. We consider the colored graph I with vertex set $\{0,1\}$, vertex colors $c_I(0) = c_I(1) = 1$ and edge color $c_I(0,1) = 1$.

Since $c_{\mathbf{m}}(0) = 1$ and $c_{\mathbf{m}}(0,1) = 1$ by the definition of $c_{\mathbf{m}}$, we can define a reduction f from I to $G_{\mathbf{m}}$ by

$$f(0) = f(1) = 0$$
$$f(0,1) = (0,1).$$

Since $l \in \mathrm{Even}_{\mathbf{n}}$, we have $c_{\mathbf{n}}(2l) = 1$ and $c_{\mathbf{n}}(2l, 2l+1) = 1$ by the definition of $c_{\mathbf{n}}$. Therefore, we can define a reduction g from I to $G_{\mathbf{n}}$ by

$$g(0) = g(1) = 2l$$
$$g(0,1) = (2l, 2l+1).$$

Thus $\xi = (f,g)$ is an unfolding of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$. We have $x^* \in D^0_{\mathbf{m}}$ by the definition of $D^0_{\mathbf{m}}$, $F(x^*) \in D^l_{\mathbf{n}}$ by the choice of l. Hence x^* and ξ are compatible with respect to F, as witnessed by the fact that $(0,1) \sim_\xi (2l, 2l+1)$. □

Claim. *For every* r *with* $0 \leq r < r^*$*, there is some* $x \in X^{\mathbf{m},\mathbf{n}}_{2l,F}$ *with* $d(x^*, x) = r$*.*

Proof. We will write $X_{2l} = X^{\mathbf{m},\mathbf{n}}_{2l,F}$. By the previous claim, we can assume that $r > 0$. Towards a contradiction, suppose that there is no $x \in X_{2l}$ with $d(x^*, x) = r$. It follows that $X_{2l} \cap B_r(x^*) = X_{2l} \cap \bar{B}_r(x^*)$, where

$$\bar{B}_r(x^*) = \{x \in X \mid d(x^*, x) \leq r\}$$

denotes the closed ball of radius r around x^* in X. Let

$$U = X_{2l} \cap B_r(x^*) = X_{2l} \cap \bar{B}_r(x^*).$$

We have that $X_{2l} \cap B_r(x^*)$ is open by Lemma 3.5 and $X_{2l} \cap \bar{B}_r(x^*)$ is relatively closed in X^* by Lemma 3.6 and hence closed in X. Thus U is a subset of X^* with $x^* \in X^*$ that is both open and closed in X. However, this contradicts the choice of x^* and X^* in the beginning of the proof. □

Claim. $\bigcup_{(f,g)\in\mathbb{U}_{2l}} \operatorname{ran}(f) = \mathbb{N}$.

Proof. Suppose that $j \geq 0$ and r is chosen with $r_{2j} < r < r_{2j+1}$. By the previous claim, there is some $x \in X_{2l}$ with $d(x^*, x) = r$. In particular, we have $x \in D^j_{\mathbf{n}}$ by the definition of $D^j_{\mathbf{n}}$.

By the definition of $X_{2l} = X^{\mathbf{m},\mathbf{n}}_{2l,F}$, there is some unfolding $\xi = (f, g) \in \mathbb{U}^{\mathbf{m},\mathbf{n}}_{2l}$ with $x \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$. Thus x and ξ are compatible with respect to F. Together with the fact that $x \in D^j_{\mathbf{n}}$ and by the definition of compatibility, this implies that $2j \in \mathrm{ran}(f)$. Since moreover $0 \in \mathrm{ran}(f)$ by the definition of $\mathbb{U}^{\mathbf{m},\mathbf{n}}_{2l}$ and since $\mathrm{ran}(f)$ is an interval by the definition of reductions, it follows that $\{0, \ldots, 2j\} \subseteq \mathrm{ran}(f)$. □

The claim now follows from Lemma 2.13. □

We are now ready to complete the proof of the main result.

Theorem 2.15. *For any metric space (X, d) of positive dimension, there are uncountably many Borel subsets of (X, d) that are pairwise incomparable with respect to continuous reducibility.*

Proof. Let A denote the subset of the Baire space $\mathbb{N}^{\mathbb{N}}$ consisting of the sequences $\mathbf{n}$ beginning with 0 such that both $\mathbf{n}$ and $\Delta\mathbf{n}$ are strictly increasing. Since A is a closed subset of $\mathbb{N}^{\mathbb{N}}$, it is itself a Polish space. Moreover, we consider the equivalence relation E on A defined by

$$(\mathbf{m}, \mathbf{n}) \in E \Longleftrightarrow (\Delta\mathbf{m}, \Delta\mathbf{n}) \in E_{\mathrm{tail}}.$$

Then E is a Borel equivalence relation on A whose equivalence classes are countable. Therefore, it is easy to check and follows from standard results (see e.g. [Jec03, Lemma 32.2] and [Kec95, Theorem 8.41]) that there is a perfect subset of A whose elements are pairwise E-inequivalent. The claim now follows from Lemma 2.14. □

Moreover, we can now easily obtain the following application of the main result.

Theorem 2.16. *Suppose that (X, d) is a Polish metric space. Then the following conditions are equivalent.*

(a) X has dimension 0.
(b) $\mathsf{SLO}_{(X,\tau)}$ holds.
(c) The Wadge order on the Borel subsets of X is a well-quasiorder.
(d) There are at most two pairwise incomparable Borel subsets of X.
(e) There are at most countably many pairwise incomparable Borel subsets of X.

Proof. It is well-known that any Polish metric space (X, d) of dimension 0 is homeomorphic to the set of branches $[T]$ of some subtree T of ${}^{<\omega}\omega$. Since the proof of Wadge's lemma (see e.g. [And07, Section 2.3]) works for these spaces, it follows from Borel determinacy that $\mathsf{SLO}_{(X,d)}$ holds. The remaining implications follow immeidately from Theorem 2.15. □

3. Auxiliary results

In this section, we complete the missing proofs from the previous section.

3.1. The proof that the compatibility range is open and closed.

Before we prove Lemma 3.5 and Lemma 3.6, we will prove two auxiliary results.

Definition 3.1. Suppose that $x \in X^*$. A subset U of X^* is called *small at x* if $x \in U$ and the following conditions hold for all $i, j \in \mathbb{N}$ with $i < j$ and the open interval $A = (i, j)$ in $\mathbb{R}$.

(a) If $x \in C_A$, then $U \subseteq C_A$.
(b) If $F(x) \in C_A$, then $U \subseteq C_A$.

The next lemma is almost immediate from the previous definition.

Lemma 3.2. *Suppose that $F\colon X \to X$ is continuous and $x, F(x) \in X^*$. Then there is an open subset U of X^* that is small at x.*

Proof. First, we note that C_A as given in Definition 2.1 is an open subset of X whenever A is an open subset of $\mathbb{R}_{\geq 0}$. Moreover, by the definition of C_A, X^* is the union of all sets $C_{(i,j)}$, where $i, j \in \mathbb{N}$ and $i < j$. Hence we can associate to every $x \in X^*$ the smallest interval $A_x = (i, j)$ with $i, j \in \mathbb{N}$, $i < j$ and $x \in C_{(i,j)}$. Then $U = C_{A_x} \cap F^{-1}[C_{A_{F(x)}}]$ is small at x. Moreover U is open, since F is continuous. □

Let $\Gamma = C_A$, where $A = \{r_t \mid t \geq 1\}$. The next lemma reduces the later proofs to the case that $x, F(x) \in \Gamma$.

Lemma 3.3. *Suppose that* $\mathbf{m}$, $\mathbf{n}$ *are as above,* $F\colon X \to X$ *is continuous and* $x, F(x) \in X^*$ *with* $x \notin \Gamma$ *or* $F(x) \notin \Gamma$. *If* $y \in U$, *where* U *is an open subset of* X^* *that is small at* x, *then*

$$x \in X^{\mathbf{m},\mathbf{n}}_{\xi,F} \Longleftrightarrow y \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$$

for all unfoldings ξ.

Proof. We will distinguish the following cases, which depend on the values of x and $F(x)$.

Case. $x, F(x) \notin \Gamma$.

Proof. We can assume that $x \in D_{\mathbf{m}}$, since the case that $x \in E_{\mathbf{m}}$ is symmetric. Since $x \in D_{\mathbf{m}} \setminus \Gamma$, we have $x \in C_{(2j,2j+1)}$ for some $j \geq 0$, and by the assumption that U is small at x, this implies that $U \subseteq C_{(2j,2j+1)}$.

Since $x \in D_{\mathbf{m}} = F^{-1}[D_{\mathbf{m}}]$, we have $F(x) \in D_{\mathbf{n}}$ and thus $F(x) \in D_{\mathbf{n}} \setminus \Gamma$ by the case assumption. Hence $F(x) \in C_{(2k,2k+1)}$ for some $k \geq 0$, and by the assumption that U is small at x, this implies that $F[U] \subseteq C_{(2k,2k+1)}$.

The equivalence follows from the inclusions $U \subseteq C_{(2j,2j+1)}$ and $F[U] \subseteq C_{(2k,2k+1)}$. □

Case. $x \notin \Gamma$ *and* $F(x) \in \Gamma$.

Proof. We can assume that $x \in D_{\mathbf{m}}$, since the case that $x \in E_{\mathbf{m}}$ is symmetric. Since $x \in D_{\mathbf{m}} \setminus \Gamma$, we have $x \in C_{(2j,2j+1)}$ for some $j \geq 0$, and by the assumption that U is small at x, this implies that $U \subseteq C_{(2j,2j+1)}$.

By the assumption that $F(x) \in \Gamma$, we can assume that $d(x^*, F(x)) = r_i$ and $i = 2k$ for some $k \geq 1$, since the case that $i = 2k+1$ for some $k \geq 0$ is symmetric. Since U is small at x and $d(x^*, F(x)) = 2k$, we have $F[U] \subseteq C_{(2k-1,2k+1)}$, and since moreover

$$U \subseteq C_{(2j,2j+1)} \subseteq D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}],$$

we have

$$F[U] \subseteq C_{(2k-1,2k+1)} \cap D_{\mathbf{n}} = C_{[2k,2k+1)}.$$

(Note that in this case $k \in \mathrm{Even}_{\mathbf{n}}$, since $d(x^*, F(x)) = 2k$ and $F(x) \in D_{\mathbf{n}}$.)

The equivalence follows from the inclusions $U \subseteq C_{(2j,2j+1)}$ and $F[U] \subseteq C_{[2k,2k+1)}$. □

Case. $x \in \Gamma$ *and* $F(x) \notin \Gamma$.

Proof. We can assume that $x \in D_{\mathbf{m}}$, since the case $x \in E_{\mathbf{m}}$ is symmetric. Since $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, this implies that $F(x) \in D_{\mathbf{n}}$, and hence $F(x) \in D_{\mathbf{n}} \setminus \Gamma$ by the assumption. Then $F(x) \in C_{(2k,2k+1)}$ for some $k \geq 0$, and since U is small at x, this implies that $F[U] \subseteq C_{(2k,2k+1)}$.

By the assumption that $x \in \Gamma$, we can assume that $d(x^*, x) = r_i$ and $i = 2j$ for some $j \geq 1$, since the case that $i = 2j+1$ for some $j \geq 0$ is symmetric. Since U is small at x and $d(x^*, x) = 2j$, we have $U \subseteq C_{(2j-1,2j+1)}$, and since moreover

$$F[U] \subseteq C_{(2k,2k+1)} \subseteq D_{\mathbf{n}}$$

and $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, we have $U \subseteq D_{\mathbf{m}}$. Hence

$$U \subseteq C_{(2j-1,2j+1)} \cap D_{\mathbf{m}} = C_{[2j,2j+1)}.$$

(Note that in this case $j \in \mathrm{Even}_{\mathbf{m}}$, since $d(x^*, x) = 2j$ and $x \in D_{\mathbf{m}}$.)

The equivalence follows from the inclusions $U \subseteq C_{[2j,2j+1)}$ and $F[U] \subseteq C_{(2k,2k+1)}$. □

Since we have covered all cases, the conclusion of Lemma 3.3 follows. □

Whenever ξ, ξ^* are unfoldings, we say that ξ^* *extends* ξ if every pair or vertices or edges that appears in ξ also appears in ξ^*.

Lemma 3.4. *Suppose that* $k, l > 0$ *and* ξ *is an unfolding of* $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ *with*

$$(k, k+1) \sim_\xi (l, l+1)$$

and $c_{\mathbf{m}}(k) = c_{\mathbf{n}}(l)$. *Then there is some unfolding* ξ^* *of* $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ *such that* $\sim_{\xi^*}$ *extends* $\sim_\xi$ *and*

$$(k-1, k) \sim_{\xi^*} (l-1, 1).$$

Proof. Suppose that $\mathrm{dom}(\xi) = [u, u^*]$. By the assumption and by the definition of $\sim_\xi$, there is some $j \in \mathrm{dom}(\xi)$ with $f(j, j+1) = (k, k+1)$ and $g(j, j+1) = (l, l+1)$.

We define the following coloring on the vertices in the interval $[u-1, u^*+1]$ and on the edges between adjacent vertices, and thereby define a colored graph I.

$$c_I(i) = \begin{cases} c_\xi(i+1) & \text{if } i \leq j-1 \\ c_{\mathbf{m}}(k) = c_{\mathbf{n}}(l) & \text{if } i \in \{j, j+1\} \\ c_\xi(i-1) & \text{if } i \geq j+2 \end{cases}$$

$$c_I(i, i+1) = \begin{cases} c_\xi(i+1, i+2) & \text{if } i \leq j-1 \\ 1 - c_{\mathbf{m}}(k, k+1) = 1 - c_{\mathbf{n}}(l, l+1) & \text{if } i = j \\ c_\xi(i-1, i) & \text{if } i \geq j+1 \end{cases}$$

We further define the following maps f^*, g^* on the vertices and edges of I.

$$f^*(i) = \begin{cases} f(i)+1 & \text{if } i \leq j-1 \\ k & \text{if } i \in \{j, j+1\} \\ f(i)-1 & \text{if } i \geq j+2 \end{cases}$$

$$g^*(i) = \begin{cases} g(i)+1 & \text{if } i \leq j-1 \\ l & \text{if } i \in \{j, j+1\} \\ g(i)-1 & \text{if } i \geq j+2 \end{cases}$$

$$f^*(i, i+1) = \begin{cases} f(i+1, i+2) & \text{if } i \leq j-1 \\ (k-1, k) & \text{if } i = j \\ f(i-1, i) & \text{if } i \geq j+1 \end{cases}$$

$$g^*(i, i+1) = \begin{cases} g(i+1, i+2) & \text{if } i \leq j-1 \\ (l-1, l) & \text{if } i = j \\ g(i-1, i) & \text{if } i \geq j+1 \end{cases}$$

The statement that $\xi^* = (f^*, g^*)$ is an unfolding that extends ξ can be checked from the definitions above. □

We are now ready to prove Lemma 3.5 above, which we restate below.

Lemma 3.5. *For all* $\mathbf{m}$, $\mathbf{n}$, F *as above and* $i^* \geq 0$, $X^{\mathbf{m},\mathbf{n}}_{i^*,F}$ *is an open subset of* X^*.

Proof. Suppose that $x \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$. By Lemma 3.2, there is an open subset of X^* that is small at x, and it is sufficient to show that $U \subseteq X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

The claim follows from Lemma 3.3 if $x \notin \Gamma$ or $F(x) \notin \Gamma$. Therefore, we may assume that $x, F(x) \in \Gamma$, so that $d(x^*, x) = r_k$ and $d(x^*, F(x)) = r_l$ for some $k, l \geq 1$. We may further assume that $k = 2i$ and $l = 2j$ for some $i, j \geq 1$, since the cases that $k = 2i+1$ or $l = 2j+1$ for some $j \geq 0$ are symmetric. We may finally assume that $x \in D_{\mathbf{m}}$, since the case that $x \in E_{\mathbf{m}}$ is symmetric.

Since $d(x^*, x) = r_{2i}$, the last assumption implies that $i \in \mathrm{Even}_{\mathbf{m}}$. Since $x \in D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, we further have $F(x) \in D_{\mathbf{n}}$, and since $d(x^*, F(x)) = r_{2j}$, this implies that $j \in \mathrm{Even}_{\mathbf{n}}$. It now follows from $i \in \mathrm{Even}_{\mathbf{m}}$ and $j \in \mathrm{Even}_{\mathbf{n}}$ that

$$c_{\mathbf{m}}(k) = c_{\mathbf{m}}(2i) = c_{\mathbf{n}}(2j) = c_{\mathbf{n}}(l).$$

Since $x \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$, there is some unfolding $\xi = (f, g)$ with $(k, k+1) \sim_\xi (l, l+1)$. Now Lemma 3.4 yields an unfolding ξ^* of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ such that $\sim_{\xi^*}$ extends $\sim_\xi$ and $(k-1, k) \sim_{\xi^*} (l-1, l)$.

It is sufficient to show the following.

Claim. $U \subseteq X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

Proof. Suppose that $y \in U$ is given. Since U is small at x and $d(x^*, F(x)) = r_{2j}$, we have $F[U] \subseteq C_{(2j-1,2j+1)}$.

Case. $y \in C_{(2i-1,2i)}$.

Since $y \in C_{(2i-1,2i)} \subseteq E_{\mathbf{m}}$ and $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, we have $F(y) \in E_{\mathbf{n}}$, and moreover, since $j \in \mathrm{Even}_{\mathbf{n}}$,

$$F(y) \in C_{(2j-1,2j+1)} \cap E_{\mathbf{n}} = C_{(2j-1,2j)}.$$

We have $(2i-1,2i) \sim_{\xi^*} (2j-1,2j)$ by the choice of ξ^* and $0 \sim_{\xi^*} i^*$ since ξ^* extends ξ. It follows that $y \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

Case. $y \in C_{[2i,2i)}$.

In this case, it follows from

$$y \in C_{[2i,2i+1)} \subseteq D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}],$$

that $F(y) \in D_{\mathbf{n}}$, and since moreover $j \in \mathrm{Even}_{\mathbf{n}}$, we have

$$F(y) \in C_{(2j-1,2j+1)} \cap D_{\mathbf{n}} = C_{[2j,2j+1)}.$$

We have $(2i,2i+1) \sim_{\xi^*} (2j,2j+1)$ by the choice of ξ and $0 \sim_{\xi^*} i^*$ since ξ^* extends ξ. It follows that $y \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$. □

The previous subclaim completes the proof of Lemma 3.5. □

We are now ready to prove Lemma 3.6 above, which we restate below.

Lemma 3.6. *For all* $\mathbf{m}$, $\mathbf{n}$, F *as above and* $i^* \geq 0$, $X^{\mathbf{m},\mathbf{n}}_{i^*,F}$ *is a relatively closed subset of* X^*.

Proof. Suppose that $x = \lim_{s \geq 0} x_s$ for some sequence $\mathbf{x} = \langle x_s \mid s \geq 0 \rangle$ with $x_s \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$ for all $s \geq 0$. It is sufficient to show that $x \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

By Lemma 3.2, there is an open subset U of X^* that is small at x. First suppose that $x \notin \Gamma$ or $F(x) \notin \Gamma$. Since $x = \lim_{s \geq 0} x_s$ and U is open and contains x, there is some $s \geq 0$ with $x_s \in U$. Since moreover $x_s \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$, there is some unfolding ξ with $x_s \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$ and $0 \sim^{\mathbf{m},\mathbf{n}}_{\xi} i^*$. By Lemma 3.3, this implies that $x \in X^{\mathbf{m},\mathbf{n}}_{\xi,F} \subseteq X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

We now suppose that $x, F(x) \in \Gamma$. Then $d(x^*,x) = r_k$ and $d(x^*,F(x)) = r_l$ for some $k, l \geq 1$. We can assume that $k = 2i$ and $l = 2j$ for some $i, j \geq 1$, since the cases that $k = 2i+1$ or $l = 2j+1$ for some $j \geq 0$ are symmetric. Since $d(x^*,x) = r_k = r_{2i}$, this implies that $i \in \mathrm{Odd}_{\mathbf{m}}$. Since moreover $x \in E_{\mathbf{m}}$, $F(x) \in X^*$ and $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, we obtain $F(x) \in E_{\mathbf{n}}$, and since $d(x^*,F(x)) = r_{2j}$, this implies that $j \in \mathrm{Odd}_{\mathbf{n}}$.

Since U is small at x, $d(x^*,x) = r_{2i}$ and $d(x^*,F(x)) = r_{2j}$, it follows that $U \subseteq C_{(2i-1,2i+1)}$ and $F[U] \subseteq C_{(2i-1,2i+1)}$. Moreover, since U is an open set containing x, there is some s_0 with $x_s \in U$ for all $s \geq s_0$.

Case. $x_s \in D_{\mathbf{m}}$ *for some* $s \geq s_0$.

Since $s \geq s_0$, we have $x_s \in U \subseteq C_{(2i-1,2i+1)}$. Therefore, by the case assumption and the fact that $i \in \mathrm{Odd}_{\mathbf{m}}$, we have $x_s \in C_{(2i-1,2i+1)} \cap D_{\mathbf{m}} = C_{(2i,2i+1)}$.

Since $x_s \in D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, we have $F(x_s) \in D_{\mathbf{n}}$, and since moreover $s \geq s_0$, it follows that $F(x_s) \in F[U] \subseteq C_{(2j-2,2j+1)}$. Using the fact that $j \in \mathrm{Odd}_{\mathbf{n}}$, we obtain

$$F(x_s) \in C_{(2j-1,2j+1)} \cap D_{\mathbf{n}} = C_{(2j,2j+1)}.$$

By the assumption that $x_s \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$ and by the definition of $X^{\mathbf{m},\mathbf{n}}_{i^*,F}$, there is some unfolding ξ of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ with $x_s \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$ and $0 \sim_{\xi} i^*$.

We have already argued that $x_s \in C_{(2i,2i+1)}$, $F(x_s) \in C_{(2j,2j+1)}$ and $x_s \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$, and therefore $(2i,2i+1) \sim_{\xi} (2j,2j+1)$ by the definition of $\sim_{\xi}$.

Since $i \in \mathrm{Odd}_{\mathbf{m}}$ and $j \in \mathrm{Odd}_{\mathbf{n}}$, we have $c_{\mathbf{m}}(2i) = c_{\mathbf{n}}(2j)$. Therefore, Lemma 3.4 yields some unfolding ξ^* such that $\sim_{\xi^*}$ extends $\sim_{\xi}$ and $(2i-1,2i) \sim_{\xi^*} (2j,2j+1)$. It follows that $x \in X^{\mathbf{m},\mathbf{n}}_{\xi,F} \subseteq X^{\mathbf{m},\mathbf{n}}_{i^*,F}$.

Case. $x_s \in E_{\mathbf{m}}$ *for some* $s \geq s_0$.

Since $s \geq s_0$, we have $x_s \in U \subseteq C_{(2i-1,2i+1)}$. Therefore, by the case assumption and the fact that $i \in \mathrm{Odd}_{\mathbf{m}}$,

$$x_s \in C_{(2i-1,2i+1)} \cap E_{\mathbf{m}} = C_{(2i-1,2i+1]}.$$

Since $x_s \in E_{\mathbf{m}}$, $F(x) \in X^*$ and $D_{\mathbf{m}} = F^{-1}[D_{\mathbf{n}}]$, it follows that $F(x_s) \in E_{\mathbf{n}}$, and since moreover $s \geq s_0$, we have $F(x_s) \in F[U] \subseteq C_{(2j-1,2j+1)}$. Since $i \in \mathrm{Odd}_{\mathbf{m}}$, it follows that

$$F(x_s) \in C_{(2j-1,2j+1)} \cap E_{\mathbf{n}} = C_{(2j-1,2j]}.$$

By the assumption that $x_s \in X^{\mathbf{m},\mathbf{n}}_{i^*,F}$ and by the definition of $X^{\mathbf{m},\mathbf{n}}_{i^*,F}$, there is some unfolding ξ of $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ with $x_s \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$ and $0 \sim_\xi i^*$.

We have already argued that $x_s \in C_{(2i-1,2i]}$, $F(x_s) \in C_{(2j-1,2j]}$ and $x_s \in X^{\mathbf{m},\mathbf{n}}_{\xi,F}$, and therefore $(2i-1,2i) \sim_\xi (2j-1,2j)$ by the definition of $\sim_\xi$. It follows that $x \in X^{\mathbf{m},\mathbf{n}}_{\xi,F} \subseteq X^{\mathbf{m},\mathbf{n}}_{i^*,F}$. □

3.2. **The proof of $\mathrm{E}_{\mathrm{tail}}$-equivalence.** In this section, we give a proof of Lemma 2.13, but before we can do this, we will prove two auxiliary results.

Lemma 3.7. *Suppose that* $\mathbf{m}$, $\mathbf{n}$ *are as above,* $s, t \geq 0$ *and* $\xi = (f, g)$ *is an unfolding of* $G_{\mathbf{m}}$, $G_{\mathbf{n}}$ *with*

$$\mathrm{ran}(f) \subseteq [2m_s, 2m_{s+1} - 1]$$

$$\mathrm{ran}(g) \subseteq [2n_t, 2n_{t+1} - 1].$$

Then

$$f(i) - f(j) = (-1)^{s+t}(g(i) - g(j)) \tag{$*_{i,j}$}$$

for all $i, j \in \mathrm{dom}(\xi)$.

Proof. Suppose that f is a reduction from I to $G_{\mathbf{m}}$ and g is a reduction from I to $G_{\mathbf{n}}$, where I is a finite colored graph. To prove the claim, we can assume that $i \leq j$, since the case $i \geq j$ is symmetric. We now fix i and prove the formula $(*_{i,j})$ by induction on $j \geq i$. Therefore, we assume that the formula holds for some $j \geq i$ with $j + 1 \in \mathrm{dom}(\xi)$.

Case. $f(j+1) = f(j)$.

Since f, g are reductions, the case assumption implies that

$$c_{\mathbf{n}}(g(j+1)) = c_I(j+1) = c_{\mathbf{m}}(f(j+1)) = c_{\mathbf{m}}(f(j)) = c_I(j) = c_{\mathbf{n}}(g(j)).$$

Since the colors of vertices alternate in $[2n_t, 2n_{t+1} - 1]$ by the definition of $G_{\mathbf{n}}$, and since g is a reduction, this implies that $g(j+1) = g(j)$. Thus the formula $(*_{i,j+1})$ follows immediately from the formula $(*_{i,j})$ that is given by the induction hypothesis.

Case. $f(j+1) \neq f(j)$.

We can assume that $f(j+1) > f(j)$, since the case that $f(j+1) < f(j)$ is symmetric. Using this assumption and the fact that f is a reduction, it follows that $f(j+1) = f(j) + 1$.

We can further assume that s, t are even and that $f(j)$ is even, because the remaining cases are symmetric. Since f, g are reductions, s is even, $f(j)$ is even and by the definition of $G_{\mathbf{m}}$, we have

$$c_{\mathbf{n}}(g(j)) = c_I(j) = c_{\mathbf{m}}(f(j)) = 1 \tag{1}$$

$$c_{\mathbf{n}}(g(j+1)) = c_I(j+1) = c_{\mathbf{m}}(f(j+1)) = c_{\mathbf{m}}(f(j)+1) = 0 \tag{2}$$

$$c_{\mathbf{n}}(g(j), g(j+1) = c_I(j, j+1) = c_{\mathbf{m}}(f(j), f(j+1)) = c_{\mathbf{m}}(f(j), f(j)+1) = 1. \tag{3}$$

Since t is even and by the definition of $G_{\mathbf{n}}$, the equations (1) and (2) imply that $g(j)$ is even and $g(j+1)$ is odd. Finally, since we argued that $g(j)$ is even, since t is even and by the definition of $G_{\mathbf{n}}$, the equation (3) implies that $g(j+1) = g(j) + 1$.

Since s, t are assumed to be even, $(-1)^{s+t} = 1$. Thus the formula $(*_{i,j+1})$ follows immediately from the statements $f(j+1) = f(j) + 1$ and $g(j+1) = g(j) + 1$. □

By keeping the range of f restricted to a single interval but allowing g to range over several intervals, we obtain the following variant of the previous lemma.

Lemma 3.8. *Suppose that* $\mathbf{m}$, $\mathbf{n}$ *are as above and* $\xi = (f, g)$ *is an unfolding of* $G_{\mathbf{m}}$, $G_{\mathbf{n}}$. *Moreover, suppose that* $a \in \mathrm{dom}(\xi)$, $s, t \geq 0$ *and* $w \geq t$ *are such that*

$$\mathrm{ran}(f) \subseteq [2m_s, 2m_{s+1} - 1]$$
$$\mathrm{ran}(g) \subseteq [g(a), 2n_w]$$
$$g(a) \in [2n_t, 2n_{t+1} - 1].$$

Let

$$b_u = \begin{cases} g(a) & \text{if } u = t \\ 2n_u & \text{if } t < u \leq w, \end{cases}$$

$$p_u = (b_{u+1} - b_u) - 1$$

for $t \leq u < w$ *and*

$$\Sigma_v = \sum_{t \leq u < v} (-1)^u p_u.$$

for $t \leq v \leq w$.

If $i \in \mathrm{dom}(\xi)$, *let* v, j *be unique with* $t \leq v < w$, $j \leq p_v$ *and* $g(i) = b_v + j < b_{v+1}$. *Then*

$$(*_i) \qquad f(i) = f(a) + (-1)^s(\Sigma_v + (-1)^v j)$$

Proof. Suppose that f is a reduction from I to $G_{\mathbf{m}}$ and g is a reduction from I to $G_{\mathbf{n}}$, where I is a finite colored graph.

We prove the formula $(*_i)$ by induction on $i \geq a$. The proof for $i \leq a$ is symmetric. Let $a^* = \max(\mathrm{dom}(\xi))$. Suppose that the formula $(*_i)$ holds for some i with $a \leq i < a^*$.

We partition $[a, a^*]$ into maximal subintervals $[a_0, a_0^*], \ldots, [a_l, a_l^*]$ such that for all $j < l$, there is some u with $t \leq u < w$ and

$$\mathrm{ran}(g{\upharpoonright}[a_j, a_{j+1}]) \subseteq [b_u, b_{u+1}].$$

We can assume that the intervals are ordered such that $a_j < a_{j+1}$ for all $j < l$.

Case 1. $i = a_k^*$ *for some* $k < l$.

By the choice of the subintervals and since $i < a$, we have $i + 1 = a_k^* + 1 = a_{k+1}$ and there is some v with $t \leq v + 1 < w$ and

$$(1) \qquad \{g(i), g(i+1)\} = \{g(a_k^*), g(a_{k+1})\} = \{b_{v+1} - 1, b_{v+1}\}.$$

Since $c_{\mathbf{n}}(b_{v+1} - 1) = c_{\mathbf{n}}(b_{v+1})$ by the definition of $G_{\mathbf{n}}$, it follows from the equation (1) and the fact that f, g are reductions that

$$(2) \qquad c_{\mathbf{m}}(f(i)) = c_I(i) = c_{\mathbf{n}}(g(i)) = c_{\mathbf{n}}(g(i+1)) = c_I(i+1) = c_{\mathbf{m}}(f(i+1)).$$

Since the colors of vertices in $[2m_s, 2m_{s+1} - 1]$ alternate by the definition of $G_{\mathbf{m}}$ and since $\mathrm{ran}(f)$ is contained in this interval by assumption, it follows from the equation (2) that $f(i) = f(i+1)$.

Since $b_{v+1} - 1 = b_v + p_v$ and $\Sigma_v + (-1)^v p_v = \Sigma_{v+1}$, it follows from the equation (1) that the formulas $(*_i)$ and $(*_{i+1})$ yield the same value. Since we argued that $f(i) = f(i+1)$ and the formula $(*_i)$ holds by the induction hypothesis, this implies that $(*_{i+1})$ holds.

Case 2. $i \in [a_k, a_k^*)$ *for some* $k \leq l$.

Suppose that $\mathrm{ran}(g{\upharpoonright}[a_k, a_k^*])) \subseteq [b_u, b_{u+1} - 1]$, where $t \leq u < w$.

Subcase 1. $c_I(i) = c_I(i+1)$.

By the subcase assumption and since f, g are reductions, we have

$$(1) \qquad c_{\mathbf{m}}(f(i)) = c_I(i) = c_I(i+1) = c_{\mathbf{m}}(f(i+1)$$

$$(2) \qquad c_{\mathbf{n}}(g(i)) = c_I(i) = c_I(i+1) = c_{\mathbf{n}}(g(i+1)$$

Since the colors of vertices alternate in $[2m_s, 2m_{s+1} - 1]$ by the definition of $G_{\mathbf{m}}$ and in $[b_u, b_{u+1}]$ by the definition of $G_{\mathbf{n}}$, it follows that $f(i+1) = f(i)$ and $g(i+1) = g(i)$.

Thus, the formula $(*_{i+1})$ follows immediately from the formula $(*_i)$, which holds by the induction hypothesis.

Subcase 2. $c_I(i) \neq c_I(i+1)$.

By the subcase assumption and since f, g are reductions, we have

$$c_{\mathbf{m}}(f(i)) = c_I(i) \neq c_I(i+1) = c_{\mathbf{m}}(f(i+1)) \tag{1}$$

$$c_{\mathbf{n}}(g(i)) = c_I(i) \neq c_I(i+1) = c_{\mathbf{n}}(g(i+1)) \tag{2}$$

$$c_{\mathbf{m}}(f(i), f(i+1)) = c_I(i, i+1) = c_{\mathbf{n}}(g(i), g(i+1)). \tag{3}$$

It follows from the equations (1) and (2) that $f(i)$, $f(i+1)$ are adjacent vertices in $G_{\mathbf{m}}$ and $g(i)$, $g(i+1)$ are adjacent vertices in $G_{\mathbf{n}}$.

We can assume that $f(i+1) = f(i)+1$, since the case that $f(i+1) = f(i)-1$ is symmetric. We can further assume that s, t and $f(a)$ are even, since the other cases are symmetric. Since

$$c_{\mathbf{n}}(g(a)) = c_I(a) = c_{\mathbf{m}}(f(a)) = 1$$

by the assumption and since t is even, this implies that $g(a)$ is even.

Now let v, j be unique with $t \leq v < w$, $j \leq p_v$ and $g(i) = b_v + j < b_{v+1}$. We can assume that v and j are even, since the other cases are symmetric. Since v is even, the induction hypothesis states that

$$f(i) = f(a) + \Sigma_v + (-1)^v j = f(a) + \Sigma_v + j,$$

where

$$\Sigma_v = \sum_{t \leq u < v} (-1)^u p_u.$$

We already argued above that $b_t = g(a)$ is even. Since $b_{t+1} = 2n_{t+1}$ is also even, $p_t = (b_{t+1} - b_t) - 1$ is odd. Since $p_u = (b_{u+1} - b_u) - 1 = (2n_{u+1} - 2n_u) - 1$ is odd for all u with $t \leq u < w$ and since v is even, Σ_v is even. Since moreover $f(a)$ and j are even by the assumption, $f(i)$ is even.

By the assumption that s is even, the fact that $f(i)$ is even and the definition of $G_{\mathbf{m}}$, it follows that

$$c_{\mathbf{n}}(g(i)) = c_I(i) = c_{\mathbf{m}}(f(i)) = 1.$$

Since v is even, $g(i) \in [2n_v, 2n_{v+1} - 1]$ by the choice of v and by the definition of $G_{\mathbf{n}}$, this implies that $g(i)$ is even.

Finally, since $f(i)$, $g(i)$ are even and $f(i+1) = f(i)+1$, it follows from the equations (1), (2) and (3) above that $g(i+1) = g(i)+1$. Thus

$$g(i+1) = g(i) + 1 = b_v + (j+1) < b_{v+1}.$$

Moreover, since v is even, the formula $(*_i)$, given by the induction hypothesis, yields

$$f(i+1) = f(i) + 1 = f(a) + \Sigma_v + (-1)^v j + 1 = f(a) + \Sigma_v + (-1)^v (j+1)$$

and thus $(*_{i+1})$ holds. □

We are now ready to prove Lemma 2.13 above, which we restate below.

Lemma 3.9. *Suppose that* $\mathbf{m}$, $\mathbf{n}$ *are strictly increasing sequences of natural numbers beginning with* 0 *such that also* $\Delta\mathbf{m}$, $\Delta\mathbf{n}$ *are strictly increasing. Moreover, suppose that*

$$\bigcup_{(f,g) \in \mathbb{U}_l^{\mathbf{m},\mathbf{n}}} \operatorname{ran}(f) = \mathbb{N}$$

for some $l \geq 0$. *Then* $\Delta\mathbf{m}$, $\Delta\mathbf{n}$ *are* E_{tail}*-equivalent.*

Proof. Let $p_u = (m_{u+1} - m_u) - 1$ and $q_v = (n_{v+1} - n_v) + 1$ for $u, v \geq 0$. Moreover, suppose that

$$l \in [2n_t, 2n_{t+1} - 1]$$

and let

$$\Phi = \{p_u \mid u \geq 0,\ p_u > p_0, q_t\}$$
$$\Psi = \{q_v \mid v \geq 0,\ q_v > p_0, q_t\}.$$

Claim. *If* $u \geq 0$, $p_u \in \Phi$ *and* v *is least with* $p_u \leq q_v$, *then* u, v *have the same parity, i.e. both are even or both are odd.*

Proof. Let

$$a = 2m_u + p_u = 2m_{u+1} - 1$$

and

$$b = 2n_v + p_u.$$

By the assumptions, there is some unfolding $\xi = (f, g) \in \mathbb{U}_l^{\mathbf{m},\mathbf{n}}$ with $a \in \mathrm{ran}(f)$. Suppose that f is a reduction from I to $G_{\mathbf{m}}$ and g is a reduction from I to $G_{\mathbf{n}}$, where I is a finite colored graph.

Since $\xi \in \mathbb{U}_l^{\mathbf{m},\mathbf{n}}$, there is some $j_0 \in \mathrm{dom}(f)$ with $f(j_0) = 0$ and $g(j_0) = l$. Moreover, since $a \in \mathrm{ran}(f)$, there is some $k \in \mathrm{dom}(f)$ with $f(k) = a$. We can assume that $k > j_0$, since the case that $k < j_0$ is symmetric.

The next subclaim follows almost immediately from the definitions.

Subclaim. *$f(j_0) < 2m_u \leq a$ and $g(j_0) < 2n_v \leq b$.*

Proof. For the first claim, the assumption $p_u \in \Phi$ implies that $p_0 < p_u$. Since $\Delta\mathbf{m}$ is strictly increasing, the last inequality implies that $0 < u$ and hence $0 \leq 2n_1 - 1 < 2n_u$. Therefore,

$$f(j_0) = 0 < 2m_u \leq 2m_u + p_u = a.$$

The proof of the second claim is almost symmetric to the previous proof. The assumptions $p_u \in \Phi$ and $p_u \leq q_v$ imply that $q_t < p_u \leq q_v$. Since $\Delta\mathbf{n}$ is strictly increasing, the last inequality implies that $t < v$ and hence $l \leq 2n_{t+1} - 1 < 2n_v$. Therefore,

$$g(j_0) = l < 2n_v \leq 2n_v + p_u = b.$$

□

Let $j^* > j_0$ be least with $f(j^*) = a$ or $g(j^*) = b$. We can assume that $f(j^*) = a$, since the case that $g(j^*) = b$ is symmetric.

Subclaim. *$g(j^*) = b$.*

Proof. Suppose that $g(j^*) \neq b$. By the choice of j^* and since $g(j_0) < b$ by the previous subclaim, we have $g(j) < b$ for all $j \in [j_0, j^*]$.

Let $j < j^*$ be minimal such that $f(i) \in [2m_u, 2m_{u+1}]$ for all $i \in [j, j^*]$. Since $f(j_0) < 2m_u$ by the previous subclaim, the choice of j implies that $f(j) = 2m_u$.

We further have $q_w < p_u$ for all $w < v$ by the choice of v, and moreover $\mathrm{ran}(g{\upharpoonright}[j_0, j^*])$ is bounded strictly below $b = 2n_v + p_u$, as we argued above. Since $\Delta\mathbf{n}$ is strictly increasing, it thus follows from the sum formula in Lemma 3.8 that the maximal distance between elements of $\mathrm{ran}(f{\upharpoonright}[j, j^*])$ is strictly less than p_u. However, this contradicts the fact that $f(j) = 2m_u$ and $f(j^*) = a = 2m_u + p_u$ by the choice of j and j^*. □

Since $f(j^*) = a$ by the assumption and $g(j^*) = b$ by the previous subclaim, we have

$$c_I(a) = c_{\mathbf{m}}(f(j^*)) = c_I(j^*) = c_{\mathbf{n}}(g(j^*)) = c_I(b).$$

By the definitions of a and b in the beginning of the proof of this claim and since a, b have the same parity, it follows from the definitions of $G_{\mathbf{m}}$ and $G_{\mathbf{n}}$ that u, v have the same parity. □

We now let

$$u_0 = \min\{u \geq 0 \mid p_u \in \Phi\}$$
$$v_0 = \min\{v \geq 0 \mid q_v \in \Psi\}.$$

Claim. *$\Delta\mathbf{m}$, $\Delta\mathbf{n}$ are E_{tail}-equivalent.*

Proof. It follows immediately from the definitions of Φ and Ψ that the sequences

$$\mathbf{p} = \langle p_u \mid u \geq u_0\rangle$$
$$\mathbf{q} = \langle q_v \mid v \geq v_0\rangle$$

enumerate Φ and Ψ, respectively. By the definitions of p_u and q_v and since $\Delta\mathbf{m}$, $\Delta\mathbf{n}$ are strictly increasing by the assumption, the sequences $\mathbf{p}$, $\mathbf{q}$ are strictly increasing.

We now show that $\Phi = \Psi$. By the properties of $\mathbf{p}$, $\mathbf{q}$ that were just stated, this implies that $p_{u_0+j} = q_{v_0+j}$ for all $j \geq 0$, and hence $\Delta\mathbf{m}$, $\Delta\mathbf{n}$ are E_{tail}-equivalent, proving the claim. We will only show that $\Phi \subseteq \Psi$, since the proof of the other inclusion is symmetric.

Suppose that $u \in \Phi$ and v is least with $p_u \leq q_v$. Then u, v have the same parity by the first claim.

Subclaim. $q_v < p_{u+1}$.

Proof. Towards a contradiction, suppose that $p_{u+1} \leq q_v$. By the choice of v as least with $p_u \leq q_v$, this implies that v is also least with $p_{u+1} \leq q_v$. Therefore $u+1$, v have the same parity by the first Claim. applied to Ψ, $\mathbf{n}$, $\mathbf{m}$ instead of Φ, $\mathbf{m}$, $\mathbf{n}$. However, this contradicts the fact that u, v have the same parity. ☐

To complete the proof of the claim, it is sufficient to show that $p_u = q_v$. Towards a contradiction, suppose that $p_u < q_v$. Since $p_u < q_v < p_{u+1}$ by the previous subclaim, $w = u+1$ is least with $q_v < p_w$. This implies that $u+1$, v have the same parity by the first Claim. However, this contradicts the fact that u, v have the same parity. ☐

The statement of the previous claim completes the proof of Lemma 3.9. ☐

4. Incomparable non-definable sets

The main result suggests the question whether it is possible to construct larger families of non-definable incomparable subsets of any metric space of positive dimension. The next result shows that this is possible, if we make an additional assumption.

Theorem 4.1. *Suppose that (X, d) is a locally compact metric space of positive dimension. Then there is a (definable) injective map that takes sets of reals to subsets of X in such a way that these subsets are pairwise incomparable with respect to continuous reducibility.*

Proof. In this proof, we will use the construction that was carried out in the proof of the main theorem above. In the following, we will call a subset of a topological space that is both closed and open *closed-open*. Moreover, we will write $\mathrm{cl}(Y)$ for the closure and $d(Y)$ for the diameter of a subset Y of X.

We let A denote the set of $x \in X$ such that X has dimension 0 at x, i.e. there is a neighborhood base at x that consists solely of closed-open sets. Moreover, we let $B = X \setminus A$. Since (X, d) has positive dimension, we can choose some $x^\diamond \in B$. Since (X, d) is moreover locally compact, there is an open ball U containing $x^\diamond$ that is pre-compact, i.e. has compact closure. In particular, it follows that U is separable.

Claim. *$B \cap U$ is uncountable.*

Proof. It follows from the definition of A that its dimension is 0. If $B \cap U$ is countable, then U has dimension 0 by [HW41, Theorem II.2]. However, this would contradict the choice of $x^\diamond$. ☐

By the previous claim and since U is pre-compact, it follows from the perfect set property for closed sets [Kec95, Theorem 6.2] that there is a perfect subset C of $\mathrm{cl}(B) \cap U$ that is nowhere dense in $\mathrm{cl}(B) \cap U$.

It is then easy to construct a sequence $\mathbf{x} = \langle x_n \mid n \geq 0\rangle$ of distinct elements of $B \cap U$ and a sequence $\mathbf{r} = \langle r_n \mid n \geq 0\rangle$ of positive real numbers converging to 0 with the following properties for all distinct $i, j \in \mathbb{N}$.

(a) $\mathrm{cl}(B_{r_i}(x_i)) \subseteq U$.
(b) $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \emptyset$.
(c) $B_{r_i}(x_i) \cap C = \emptyset$.
(d) $C \subseteq \mathrm{cl}(\{x_n \mid n \geq 0\})$.

We further fix a sequence $\langle \mathbf{m}_n \mid n \geq 0\rangle$ of pairwise not E_{tail}-equivalent elements $\mathbf{m}_n$ of $\mathbb{N}^\mathbb{N}$ such that both $\mathbf{m}_n$ and $\Delta\mathbf{m}_n$ are strictly increasing. Moreover, we fix a strictly increasing sequence $\mathbf{r}_n = \langle r_{n,i} \mid i \geq 0\rangle$ with supremum r_n for each $n \geq 0$. We can now construct subsets $D^n_{\mathbf{m}_n}$ of $B_{r_n}(x_n)$ analogous to the construction in Section 2, where x^*, $\mathbf{r}$ is replaced with x_n, $\mathbf{r}_n$. Assuming that $h\colon 2^\mathbb{N} \to C$ is a bijection, we then define

$$D_I = h[I] \cup \bigcup_{n \geq 0} D^n_{\mathbf{m}_n}$$

for each subset I of $2^\mathbb{N}$.

In the following claims, we will assume that $n \geq 0$ is fixed. We let A^* denote the set of $x \in B_{r_{n,1}}(x_n)$ such that there is some subset U^* of $B_{r_{n,1}}(x_n)$ with $x \in U^*$ that is both open and closed in X. We further let $B^* = B_{r_{n,1}}(x_n) \setminus A^*$. In particular, we have $x_n \in B^*$ by the choice of x_n.

In the next claim, we call a subset U^* of $B_{r_{n,1}}(x_n)$ *bounded* if the values $d(x_n, x)$ for $x \in U^*$ are bounded strictly below $r_{n,1}$, and otherwise *unbounded.*

Claim. *Any subset U^* of B^* with $x_n \in U^*$ that is closed-open in B^* is unbounded.*

Proof. We assume towards a contradiction that U^* is bounded. Hence we can assume that $d(x_n, x) < r_n - \epsilon$ for some $\epsilon > 0$. Since U is pre-compact, there is some δ with $0 < \delta < \epsilon$ such that $d(x, y) > \delta$ for all $(x, y) \in U^* \times (B^* \setminus U^*)$.

By the definition of A^*, there is a sequence $\langle V_i \mid i \geq 0\rangle$ of subsets of $B_{r_{n,1}}(x_n)$ that are both open and closed in X with $A^* = \bigcup_{i\geq 0} V_i$. We can further assume that $\lim_{i\geq 0} d(V_i) = 0$, $d(V_i) < \frac{\delta}{2}$ and V_i is bounded for all $i \geq 0$.

If $\sim \in \{=, \leq\}$, let $S_\sim = \{x \in B^* \mid d(x, U^*) \sim \frac{\delta}{2}\}$. Moreover, let $N = \{i \in \mathbb{N} \mid V_i \cap S_= \neq \emptyset,\ d(V_i) < \frac{\delta}{2}\}$ and $V = S_\leq \setminus \bigcup_{i\in N} V_i$. It follows immediately from these definitions that V is closed-open in X and bounded. However, this contradicts the fact that $x_n \in B^*$. □

We now claim that D_I, D_J are incomparable for any two distinct subsets I, J of 2^ω. To this end, suppose that $F\colon X \to X$ is a continuous map with $D_I = F^{-1}[D_J]$.

Claim. *There is some $x \in B_{r_{n,1}}(x_n)$ with $F(x) \notin C$.*

Proof. We assume towards a contradiction that $F[B_{r_{n,1}}(x_n)] \subseteq C$. It follows that for any subset V^* of C that is closed-open in C, its pre-image $U^* = F^{-1}[V^*]$ is closed-open in $B_{r_{n,1}}(x_n)$ and therefore $U^* \cap B^*$ is closed-open in B^*. By the previous claim, $U^* \cap B^*$ is unbounded.

We can now apply this statement to arbitrarily small neighborhoods of $F(x_n)$. Since U is pre-compact, it can then be easily shown that there is some $x \in X$ with $d(x_n, x) = r_{n,1}$ and $F(x) = F(x_n)$. However, since $(x_n, x) \in D^n_{\mathbf{m}_i} \times (X \setminus D^n_{\mathbf{m}_i})$, this contradicts the fact that F is a reduction. □

We now choose some $x \in B_{r_{n,1}}(x_n)$ as in the last claim. Then $F(x) \in D^i_{\mathbf{m}_i}$ for some $i \in \mathbb{N}$. It is now possible to show as in the proof of Theorem 1.5 that this implies that $i = n$.

We have thus shown that for every $n \geq 0$, there is some $x \in B_{r_{n,1}}(x_n)$ with $F(x) \in B_{r_n}(x_n)$. Since $C \subseteq \mathrm{cl}(\{x_n \mid n \geq 0\})$ and $\mathbf{r}$ converges to 0, it then follows that $F{\restriction}C = \mathrm{id}{\restriction}C$. Moreover, since we assumed that $D_I = F^{-1}[D_J]$, it follows that $h[I] = h[J]$ and hence $I = J$. □

5. Further remarks

We state a few further observations about the the main theorem.

Remark 5.1. It follows from Theorem 1.5 that there are totally disconnected Polish spaces with uncountably many incomparable Borel subsets, for instance the complete Erdös space (see [DvM09]).

By Urysohn's metrization theorem, the conclusion of the main theorem holds for countably based regular Hausdorff spaces, but fails without this requirement by the next remark.

Remark 5.2. The conclusion of Theorem 1.5 fails for countable T_0 spaces by [MRSS15, Remark 5.35].

Moreover, the conclusion of the main theorem is optimal in the sense that the next remark prevents further embedding theorems.

Remark 5.3. There is a compact connected subspace of X of $\mathbb{R}^3$ such that any two nonempty subsets that are not equal to X are incomparable [Coo67, Theorem 11] (see the remark after Theorem 5.15 in [MRSS15]).

We finally remark that the construction in the proof of the main theorem can also be used to prove other embedding results.

Remark 5.4. If there is a partition of a metric space (X, d) into infinitely many closed-open subspaces of positive dimension, then $(\mathcal{P}(\omega), \subseteq)$ embeds into the Wadge quasi-order on the collection of Borel subsets of (X, d). This can be proved by applying the construction in the proof of the main theorem to the subspaces.

6. Questions

We conclude with some open questions. Since the sets $D_{\mathbf{n}}$ defined in the proof of the main theorem are intersections of open and closed sets, this suggests the following question.

Question 6.1. *Does the conclusion of Theorem 1.5 hold for sets A such that both A and its complement is an intersection of an open and a closed set?*

It is further open whether it is necessary for the proof of the main theorem to assume that the space is a metric space.

Question 6.2. *Does the conclusion of Theorem 1.5 hold for all regular spaces of positive dimension?*

Moreover, it is open whether the local compactness can be omitted in the construction of incomparable non-definable sets in Section 4.

Question 6.3. *Does the conclusion of Theorem 1.8 hold for all Polish spaces of positive dimension?*

In a different direction, it would be interesting to consider similar problems for functions on arbitrary metric spaces (see e.g. [Car13, Ele02]).

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